B.Sc- part-III(H). paper-III Tangent and normals 20-07-2020

problems based the eqn of the normal

Q.1 Find the abscissa of the point on the curve $ay^2=x^3$, the normal at which cuts off equal intercepts from the co-ordinates axes.

Soln: The given eqn of the point (x,y) on the curve $ay^2=x^3$.

The curve is

$\therefore$ eqn of the normal at (x,y) to the curve is

$$\frac{X-x}{-3x^2}=\frac{Y-y}{2ay}$$

$$\Rightarrow \frac{X}{3x^2}+\frac{Y}{2ay}=\left(\frac{x}{3x^2}+\frac{y}{2ay}\right)$$

$$\Rightarrow \frac{X}{3x^2\left(\frac{1}{3x}+\frac{1}{2a}\right)}+\frac{Y}{2ay\left(\frac{1}{3x}+\frac{1}{2a}\right)}=1.$$

Here, we see that not if it cuts off equal intercepts from the co-ordinate axes.

$$\frac{1}{3x^2}\left(\frac{1}{3x}+\frac{1}{2a}\right)=2ay\left(\frac{1}{3x}+\frac{1}{2a}\right)$$

$\therefore$ either $3x^2=2ay \Rightarrow \frac{1}{3x}+\frac{1}{2a}=0$

Now, if $\frac{1}{3x}+\frac{1}{2a}=0$, then $x=-\frac{2a}{3}$. But then the normal passes through the origin, for the equation becomes

$\frac{X}{3x^2}+\frac{Y}{2ay}=0$. Then, the case is trivial, the equal intercepts being zero.

Excluding this case we have $3x^2=2ay$

But $ay^2=x^3$.

Solving these two eqns

$$x^3=a\left(\frac{3x^2}{2a}\right)^2 \Rightarrow 4a^2x^3=9ax^4$$

$\therefore x^3(4a-9x)=0$ or $x=\frac{4a}{9}$

$\therefore$ either $x=0$

But when $x=0$, $y=0$. Then the normal passes through $(0,0)$ and cutting zero intercepts from each axis.

Hence, the required abscissa $=\frac{4a}{9}$

Proved

Q. 2. If the normal at any point to the curve $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ makes angle ϕ with the axis, show that its eqn is 20-07-2020

$$y\cos\phi - x\sin\phi = a\cos 2\phi$$

Soln: Equation of the curve is $f(x,y) = x^{\frac{2}{3}} + y^{\frac{2}{3}} - a^{\frac{2}{3}} = 0$

Now, eqn of the normal at (x,y) is $\frac{X-x}{f_x} = \frac{Y-y}{f_y}$

where (X, Y) are the current co-ordinates on the tangents.

or, $\frac{X-x}{\frac{2}{3}x^{-\frac{1}{3}}} = \frac{Y-y}{\frac{2}{3}y^{-\frac{1}{3}}} \Rightarrow x^{\frac{1}{3}}(X-x) = y^{\frac{1}{3}}(Y-y)$

$\therefore$ m of the normal $= \frac{x^{\frac{1}{3}}}{y^{\frac{1}{3}}} = \tan\phi$ from the eqn.

$\therefore \frac{\sin\phi}{x^{1/3}} = \frac{\cos\phi}{y^{\frac{1}{3}}} = \frac{1}{\sqrt{x^{\frac{2}{3}} + y^{\frac{2}{3}}}} = \frac{1}{a^{\frac{1}{3}}}$

$\therefore x^{\frac{1}{3}} = a^{\frac{1}{3}}\sin\phi$ and $y^{\frac{1}{3}} = a^{\frac{1}{3}}\cos\phi$.

$\therefore$ Eqn of the normal at (x,y) becomes

$$a^{\frac{1}{3}}\sin\phi(X - a\sin^3\phi) = a^{\frac{1}{3}}\cos\phi(Y - a\cos^3\phi)$$

$$\therefore X\sin\phi - Y\cos\phi = a(\sin^4\phi - \cos^4\phi)$$

$$= a(\sin^2\phi + \cos^2\phi)(\sin^2\phi - \cos^2\phi)$$

$$= a(\sin^2\phi - \cos^2\phi)$$

$$\therefore Y\cos\phi - X\sin\phi = a\cos 2\phi$$

If we take (x,y) for current co-ordinates instead of (X,Y), equation becomes

$$y\cos\phi - x\sin\phi = a\cos 2\phi.$$

Proved.